All open surfaces are leaves of simple foliations of R^3

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ABSTRACT

For any open orientable surface Σ a smooth codimension one foliation $\mathscr{F}$ on $\mathbb{R}^3$ is constructed such that:
i) $\mathscr{F}$ is without holonomy
ii) $\mathscr{F}$ has a leaf diffeomorphic to Σ.

INTRODUCTION

Recently several authors have been treating the question which open $(n-1)$-manifolds can be a leaf of a codimension one foliation on a closed n-manifold. Results in this direction are often connected with the growth type of the leaves in relation to their endsets. For instance: a leaf with polynomial growth can only have a countable set of ends of finite depth (see 1.2. for the definition of depth), or: if the dimension of the manifold is three and the foliation is without holonomy, every leaf has at most two ends. See [C-C] 1, 2, 3, 4, [G], [N], [P-S], [S]. At present it is not known if an arbitrary open surface can be a leaf of a codimension one foliation on a closed three-manifold.

In open manifolds there is more freedom, because one has available the technique of "pushing the difficulties to infinity". Using constructions as in [He] it wouldn't be too difficult to construct foliations in $\mathbb{R}^3$ which admit many different surfaces as leaves. In this paper we solve the problem under the additional restriction that all leaves are closed. Indeed, we will prove by means of an explicit construction that every open orientable surface can be a leaf of a codimension one foliation without holonomy of $\mathbb{R}^3$. The foliations constructed here will be of class C^∞, although it is actually not difficult to change them into analytic foliations (see concluding remark i)).

1. PRELIMINARIES

In this section we state a few facts needed in the sequel, some of them without proof.

1.1. *Richards' classification theorem for open surfaces*

An *open* surface is a connected non-compact 2-manifold without boundary; for reasons to be explained in 1.4. we will restrict ourselves to orientable surfaces.

For the definition of the set of *ends* (denoted with βM) of a manifold M and background information we refer to [R] or [A-S]. The set of ends βM is compact, separable, metrizable and totally disconnected. Such sets can be identified with closed subsets of Cantor's middle-third set.

In the case of surfaces one can introduce the notion of *planar* end, a planar end being one that has a neighbourhood homeomorphic to an open subset of the plane. An end is called *nonplanar* otherwise. We denote the set of non-planar ends with $\beta_1 M$; it is a closed subset of βM, possibly empty. For a surface of genus zero all ends are planar, so we will call these surfaces *planar*.

Now we can state the special version of Richards' classification theorem ([R]) that is sufficient for our purposes:

THEOREM. *Let Σ be an open orientable surface.*

1) *If $\beta_1\Sigma$ is not empty, the homeomorphism type of the pair $(\beta\Sigma, \beta_1\Sigma)$ determines Σ up to diffeomorphism. If $\beta_1\Sigma$ is empty, the finite integer genus g of Σ together with the homeomorphism type of $\beta\Sigma$ again classifies Σ up to diffeomorphism.*

2) *For every pair (X, Y) with X a compact, separable, metrizable and totally disconnected set and Y a closed subset of X there exists an open orientable surface Σ with $(X, Y) \approx (\beta\Sigma, \beta_1\Sigma)$. If Y is empty, every non-negative integer can be realised as the finite integer genus.* □

1.2. *Depth of the set of ends*

Here we define the notion of *depth* of the set of ends (compare with the notion of type as in [C-C] 4).

Let X be a compact, separable, metrizable and totally disconnected space.

For any ordinal number α we define a subset $X^{(\alpha)}$ of X as follows:

i) $X^{(0)} := X$.

ii) Assuming $X^{(\beta)}$ has been defined for any $\beta < \alpha$, we set:

$$X^{(\alpha)} := \begin{cases} \bigcap_{\beta<\alpha} X^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal} \\ \text{the set of accumulation points of } X^{(\alpha-1)} \\ \text{if } \alpha \text{ is not a limit ordinal.} \end{cases}$$

DEFINITION. The *depth* of X is the smallest ordinal α such that $X^{(\alpha)}=X^{(\alpha+1)}$. □

For example, a Cantor set is of depth zero, a finite set is of depth one, a countable set is of depth at least one etc.

The fact that depth is well-defined and some properties follow from:

LEMMA. *Let X be a compact, separable, metrizable, totally disconnected space* (*or, equivalently, a closed subset of the Cantor set*) *and let α be the depth of X. Then we have:*

1) *α is a countable ordinal.*
2) *$X^{(\alpha)} \neq \phi \Leftrightarrow X^{(\alpha)}$ is a Cantor set $\Leftrightarrow$ X is not countable.*
3) *If X is countable, α is not a limit ordinal.*

PROOF. 1) is "folklore", see for instance [Ha], p. 170.

2) if $X^{(\alpha)}$ is not empty, then $X^{(\alpha)}$ is a compact, metrizable, separable and totally disconnected space without isolated points. Another classical theorem then states that $X^{(\alpha)}$ is a Cantor set. If, on the other hand, $X^{(\alpha)}$ is empty, then $X=\bigcup_{\beta<\alpha}(X^{(\beta)}-X^{(\beta+1)})$ is a countable union of countable sets, so X is countable.

3) choose for every $\beta<\alpha$ a point $x_\beta \in X^{(\beta)}$ and consider the net $(x_\beta)_{\beta<\alpha}$. Because X is compact this net has a limit point x, which is a point of $X^{(\beta)}$ for every $\beta<\alpha$. If α is a limit ordinal, we have: $x \in X^{(\alpha)}$, a contradiction. □

1.3. *Simple foliations*

A foliation $\mathscr{F}$ on a manifold M is called *simple* if the leaf space $M/\mathscr{F}$ is a manifold (possibly non-Hausdorff). For example, all foliations of $\mathbb{R}^2$ are simple. In this case the leaves are the fibres of a fibration over a (in general not separated) one dimensional manifold (see [W]). Another example is given by foliations of $\mathbb{R}^n$ by planes $\mathbb{R}^{n-1}$. Again the leaves are the fibres of a fibration over a one-dimensional base (see [Pa]).

For foliations of $\mathbb{R}^3$ one has the following lemma:

LEMMA. Let $\mathscr{F}$ be a codimension one foliation on a simply connected manifold M. Then the following conditions are equivalent:

1) $\mathscr{F}$ is simple.
2) $\mathscr{F}$ is without holonomy.
3) $\mathscr{F}$ does not admit a closed transversal.
4) All leaves of $\mathscr{F}$ are closed.

PROOF. 1)$\Rightarrow$2) is obvious.

2)$\Rightarrow$3) uses the fact that a closed transversal is homotopic to zero, so it bounds a disk, which we may assume to be in general position w.r. to $\mathscr{F}$. Haefliger's well known argument then gives us a leaf with holonomy of infinite order.

3) $\Rightarrow$ 4) is proved by observing that a non-closed leaf meets some foliation chart in at least two plaques, so it admits a closed transversal.

4) $\Rightarrow$ 1) see [H], p. 387. $\square$

REMARK. Let $\mathscr{F}$ be a codimension one simple foliation on a simply connected manifold M and let $j: L \to M$ be the inclusion map of a leaf L into M. Then j is a proper map, so j can be extended to a map

$$\beta j: \beta L \to \beta M.$$

1.4. *Statement of results*

Any foliation of $\mathbb{R}^3$ is orientable and transversely orientable. Therefore a codimension one foliation $\mathscr{F}$ of $\mathbb{R}^3$ admits a non-zero transverse vectorfield and any leaf L of $\mathscr{F}$ is orientable. A compact leaf L of $\mathscr{F}$ bounds a compact submanifold M of $\mathbb{R}^3$; it follows that the Euler-Poincaré characteristic of L is zero, so L is a 2-torus. As a consequence of Novikov's theorem (see [No]), M contains a compact Reeb component and therefore $\mathscr{F}$ has non-trivial holonomy. In view of this, any leaf of a codimension one simple foliation of $\mathbb{R}^3$ is non-compact and orientable, i.e.: an open orientable surface. Now the aim of this paper is to prove the following converse statement:

THEOREM. *Let Σ be an open orientable surface. Then there exists a simple, smooth (C^∞) foliation $\mathscr{F}_\Sigma$ of $\mathbb{R}^3$ with a leaf diffeomorphic to Σ.*

The proof of this theorem will be in three steps: the planar case will be treated in section 3, the case of surfaces with finite positive genus in section 4 and surfaces with nonplanar ends will be dealt with in section 5. In section 2 we introduce the technical notions of "models" and "turbulization" which will be used in the constructions of the following paragraphs.

2. TECHNICALITIES

In this section we describe a family of simple foliations on $D^2 \times \mathbb{R}$, the "models", and a construction: "turbulization along a transversal of a simple foliation of $\mathbb{R}^3$".

2.1. *Models for couples* (X, Y)

First we introduce some notations.

NOTATIONS. Let $\mathscr{F}$ be a simple, codimension one foliation on $D^2 \times \mathbb{R}$ which is horizontal near $\partial D^2 \times \mathbb{R}$. Then we denote with:

i) $\pm\infty$ the two ends of $D^2 \times \mathbb{R}$.

ii) sat(∂) the union of all leaves of $\mathscr{F}$ meeting $\partial D^2 \times \mathbb{R}$.

iii) $\mathscr{N}$ a one-dimensional foliation transverse to $\mathscr{F}$ which coïncides with the vertical one on $\partial D^2 \times \mathbb{R}$.

Furthermore we denote with L_0 the leaf of $\mathscr{F}$ which contains $\partial D^2 \times \{0\}$ and with $n(u)$ the leaf of $\mathscr{N}$ through the point $u \in D^2 \times \mathbb{R}$.

Now let (X, Y) be a couple of compact, metrizable, totally disconnected sets as in 1.1 and Σ an open orientable surface with $(\beta\Sigma, \beta_1\Sigma) \approx (X, Y)$.

DEFINITION. A simple foliation $\mathscr{F}$ on $D^2 \times \mathbb{R}$, horizontal near $\partial D^2 \times \mathbb{R}$ will be called a *model for* (X, Y) and denoted with $\mathscr{M}(X, Y)$ if there exists a one-dimensional foliation $\mathscr{N}$ transverse to $\mathscr{F}$ such that the following conditions hold:

1) There exists a diffeomorphism $\psi : \Sigma_* \times \mathbb{R} \to \text{sat}(\partial)$ such that $\psi^*(\mathscr{F})$ (resp. $\psi^*\mathscr{N}$) is the horizontal (resp. vertical) foliation on $\Sigma_* \times \mathbb{R}$.

Here Σ_* denotes the surface Σ with an open disc removed.

2) If $j : L \to D^2 \times \mathbb{R}$ is the inclusion of a leaf L contained in sat(∂), then for any end ε of L we have:

$$\beta j(\varepsilon) = +\infty.$$

3) For any sequence $(u_k)_{k \in \mathbb{N}}$ in L_0 which converges to $+\infty$ in $D^2 \times \mathbb{R}$, the family $\{n(u_k) \mid k \in \mathbb{N}\}$ of leaves of the transversal foliation $\mathscr{N}$ is locally finite. $\square$

For a model $\mathscr{M}(X, \phi)$ we simply write $\mathscr{M}(X)$ and we will cal $\mathscr{M}(X)$ a *planar model for the set X*.

EXAMPLE. The foliation on $D^2 \times \mathbb{R}$ pictured in fig. 1 is a planar model $\mathscr{M}(\{1\})$. Note that in this case $D^2 \times \mathbb{R}$-sat(∂) is a foliated solid cylinder with a foliation tangent to its boundary. We call it the (*non-compact*) *Reeb component*.

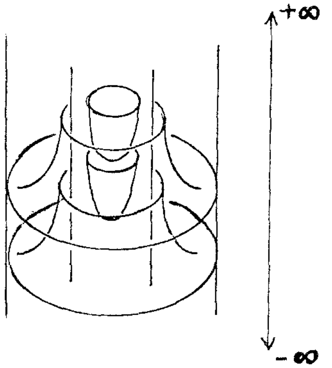

Fig. 1.

2.2. *Turbulization along a transversal*

Let $\mathscr{F}$ be a simple codimension one foliation of $\mathbb{R}^3$. According to the lemma in 1.3, a one-dimensional transverse foliation $\mathscr{N}$ has no closed leaf (i.e. a circle), so for any leaf n of $\mathscr{N}$ there exists a tubular neighbourhood W of n:

$$\phi : D^2 \times \mathbb{R} \to W$$

with $\phi(\{0\} \times \mathbb{R}) = n$ and $\phi^*\mathscr{F}$ is the horizontal foliation of $D^2 \times \mathbb{R}$.

Now take any simple foliation $\hat{\mathcal{F}}$ on $D^2 \times \mathbb{R}$ which coincides with the horizontal one near $\partial D^2 \times \mathbb{R}$. We construct a simple foliation $\mathcal{F}'$ of $\mathbb{R}^3$ by setting:

$$\mathcal{F}'|_{\mathbb{R}^3 - W} = \mathcal{F}|_{\mathbb{R}^3 - W}$$

$$\mathcal{F}'|_W \quad = (\phi^{-1})^*(\hat{\mathcal{F}})$$

and we say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a *turbulization along n modelled on* $\hat{\mathcal{F}}$. For example, a turbulization modelled on $\mathcal{M}(\{1\})$ could be called: introducing a Reeb component (a classical construction for compact manifolds). Evidently, the same construction applies when we start with a simple foliation $\mathcal{F}$ on $D^2 \times \mathbb{R}$, horizontal near $\partial D^2 \times \mathbb{R}$, provided that one takes n in the interior of $D^2 \times \mathbb{R}$.

3. PLANAR SURFACES

First we construct a planar model $\mathcal{M}(X)$ for any compact, metrizable and totally disconnected space X.

3.1. *A planar model $\mathcal{M}(X)$ for finite X*

We start with the horizontal foliation $\mathcal{F}_0$ of $D^2 \times \mathbb{R}$ (with $\mathcal{N}$ the vertical foliation). If X has p elements, we choose p points u_k in the interior of $D^2 \times \{0\}$ and make a turbulization along the verticals $n(u_k)$ $(k=1,\dots,p)$ modelled on $\mathcal{M}(\{1\})$. The resulting foliation is a planar model $\mathcal{M}(X)$ for X.

3.2. *A planar model $\mathcal{M}(X)$ for countable X*

These models will be constructed by means of a transfinite induction on the depth α of X. Our induction hypothesis is:

Induction hypothesis:

There exists a model $\mathcal{M}(X)$ for any countable, compact, metrizable and totally disconnected set X.

Indeed, $\mathcal{M}(X)$ has been constructed in 3.1 in case $\alpha = 1$, so let the depth α of X be greater than 1 and suppose the induction hypothesis has been verified for depths α' with $1 \le \alpha' < \alpha$. As X is countable, α is not a limit ordinal (1.2), so $X' := X^{(\alpha-1)}$ is a finite, non-empty set. By 3.1, there exists a model $\mathcal{M}(X')$ for X'. Now let Σ' be an open orientable planar surface with $\beta\Sigma'$ homeomorphic to X'. Specify in Σ' a closed, totally disconnected, but not necessarily compact set Z such that $\Sigma' - Z$ is homeomorphic to the planar surface Σ classified by $\beta\Sigma = X$ (this set Z can be found easily using Richards' "canonical form" for an open surface, see [R], p. 268). Write Σ' as an increasing sequence of compact submanifolds with boundary:

$$M_1 \subset M_2 \subset \dots \subset M_k \subset \dots \text{ with } \bigcup_{k=1}^{\infty} M_k = \Sigma' \text{ and } \partial M_k \cap Z = \phi \text{ for any } k,$$

and consider the collection:

$$\mathcal{C} := \{C \mid C \text{ is a component of some } M_{k+1} - \operatorname{int}(M_k) \text{ and } C \cap Z \neq \phi\}.$$

Then, for every C in $\mathscr{C}$, $C \cap Z$ is compact, so it has a well-defined depth α'_C, $1 \le \alpha'_C < \alpha$, and by our induction hypothesis there is a model $\mathscr{M}(C \cap Z)$ for $C \cap Z$.

Identify Σ' with the leaf L_0 of $\mathscr{M}(X')$ and choose for each $C \in \mathscr{C}$ a point u_C in int(C). Because $\mathscr{M}(X')$ is a model, the unique accumulation point of the set $\{u_C \mid C \in \mathscr{C}\}$ in $D^2 \times \mathbb{R}$ is $+\infty$.

So the corresponding family $\{n(u_C) \mid C \in \mathscr{C}\}$ is locally finite (again because $\mathscr{M}(X')$ is a model). Therefore we can choose mutually disjoint tubular neighbourhoods of the $n(u_C)$ and turbulize along the $n(u_C)$ on the corresponding model $\mathscr{M}(C \cap Z)$. This defines a simple foliation on $D^2 \times \mathbb{R}$ which is clearly a model $\mathscr{M}(X)$.

3.3. *A planar model $\mathscr{M}(X)$ when X is a Cantor set*

We construct a sequence $(\mathscr{M}_p)_{p \in \mathbb{N}}$ of models for finite sets as follows:

i) $\mathscr{M}_0$ is the model $\mathscr{M}(\{1\})$ introduced in 2.1.

ii) Assume that $\mathscr{M}_p$ has been constructed and is a model for some finite set X_p. According to condition 2) in the definition of models, any end ε of the leaf L_0 in $\mathscr{M}_p$ has a neighbourhood V_ε contained in $D^2 \times (p, +\infty)$. Moreover, there is a point $u_\varepsilon \in V_\varepsilon$ such that $n(u_\varepsilon) \cap D^2 \times (-\infty, p] = \phi$ (condition 3 for models). Now $\mathscr{M}_{p+1}$ is obtained from $\mathscr{M}_p$ by finitely many turbulizations along the $n(u_\varepsilon)$ modelled on $\mathscr{M}(\{1\})$. It is again a model for some finite set X_{p+1}.

If we consider the sequence $(\mathscr{M}_p)_{p \in \mathbb{N}}$ we see that, if the tubular neighbourhoods of the $n(u_\varepsilon)$ were carefully chosen, $\mathscr{M}_{p+1}$ coïncides with $\mathscr{M}_p$ outside $D^2 \times (p, \infty)$. This implies that the procedure converges to obtain a simple foliation $\mathscr{F}$ on $D^2 \times \mathbb{R}$.

It is not difficult to see that $\mathscr{F}$ is indeed a model for some set X. Finally, by construction of $\mathscr{F}$, the leaves of $\mathscr{F}$ in sat(∂) are planar open surfaces without isolated ends, or, equivalently, with a Cantor set of ends. So X is a Cantor set and $\mathscr{F}$ is a model for X.

3.4. *A planar model $\mathscr{M}(X)$ when X is not countable*

Let α be the depth of X. According to 1.2. $X' = X^{(\alpha)}$ is a Cantor set and by 3.3. there is a model $\mathscr{M}(X')$ for X'.

We now proceed exactly as in 3.2.: Let Σ' be a planar surface with X' as set of ends; choose $Z \subset \Sigma'$ totally disconnected, closed, such that $\beta(\Sigma' - Z)$ is homeomorphic to X. Then write Σ' as an increasing sequence of compact submanifolds $(M_k)_{k \in \mathbb{N}}$ s.t. $\partial M_k \cap Z = \phi$ and make for all k a turbulization in each component C of $M_{k+1} - \mathrm{int}(M_k)$ (see 3.2).

3.5. *Construction of $\mathscr{F}_\Sigma$ when Σ is planar*

Here we prove the main theorem (1.4) in the case Σ is planar. If $\Sigma = \mathbb{R}^2$, there are lots of possibilities, so suppose $\Sigma \not\approx \mathbb{R}^2$. Then we discern two cases:

1) $\beta\Sigma$ is not a Cantor set. Then $\beta\Sigma$ has an isolated point, say ε; we consider a planar model $\mathscr{M}(X)$ for $X = \beta\Sigma - \{\varepsilon\} \neq \phi$. Then $\mathscr{F}_\Sigma$ will be the restriction of $\mathscr{M}(X)$ to the interior of $D^2 \times \mathbb{R}$. Any leaf contained in sat(∂) $- \partial(D^2 \times \mathbb{R})$ is diffeomorphic to Σ, which concludes case 1).

2) $\beta\Sigma$ is a Cantor set. Then we start with the horizontal foliation on $D^2 \times \mathbb{R}$. Choose a sequence of point (u_k) in L_0 which converges to a point in ∂D^2. Now restrict to int $(D^2 \times \mathbb{R})$ and make turbulizations along the $n(u_k)$ on a model $\mathcal{M}(X)$ where X is a Cantor set. This concludes case 2) □

REMARK. In each case $\mathcal{F}_\Sigma$ has an open saturated subset U foliated as a product $\Sigma \times \mathbb{R}$ (for a suitable transverse $\mathcal{N}$).

4. SURFACES OF FINITE POSITIVE GENUS

In this section we will construct $\mathcal{F}_\Sigma$ in case Σ has finite, non-zero genus.

4.1. *The foliation $\mathcal{G}$ on $D^2 \times \mathbb{R}$ or how to increase the genus of leaves*

Let T^2_* be the torus T^2 with an open disc removed. The product $T^2_* \times \mathbb{R}$ is homeomorphic to $D^2 \times \mathbb{R}$ with two solid cylinders removed (see fig. 2). By filling in these two cylinders with two non-compact Reeb components R_0 and R_1, we obtain a simple foliation on $D^2 \times \mathbb{R}$, horizontal near $\partial D^2 \times \mathbb{R}$ with sat(∂) diffeomorphic to $T^2_* \times \mathbb{R}$. We denote it with $\mathcal{G}$.

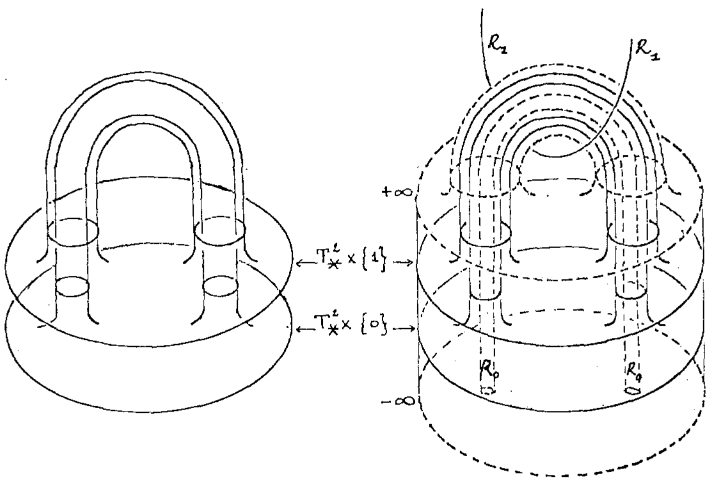

Fig. 2. The model $\mathcal{G}$.

A turbulization modelled on $\mathcal{G}$ of a simple foliation $\mathcal{F}$ along a transversal n will increase by 1 the genus of every leaf that meets n.

4.2. *Construction of $\mathcal{F}_\Sigma$ when Σ has finite genus*

Let Σ be an open orientable surface with finite genus $g > 0$ and let Σ' be the open orientable planar surface such that $\beta\Sigma' \approx \beta\Sigma$. In the foliation $\mathcal{F}_{\Sigma'}$ constructed in 3.5, there is an open saturated subset U foliated as a product $\Sigma' \times \mathbb{R}$ (for a suitable transverse foliation $\mathcal{N}$). Now let L be a leaf of $\mathcal{F}_{\Sigma'}$ in U; choose g points $u_1, \ldots, u_g$ in L and make a turbulization along the $n(u_k)$, $k = 1, \ldots, g$, modelled on $\mathcal{G}$. The resulting foliation is the desired $\mathcal{F}_\Sigma$. Again it has a saturated open subset foliated as a product $\Sigma \times \mathbb{R}$.

5. SURFACES WITH NONPLANAR ENDS

Again the preliminary step is to construct models $\mathcal{M}(X, Y)$ for pairs of compact, metrizable, totally disconnected sets where $Y \subset X$, Y closed, but this time $Y \neq \phi$.

5.1. *A model $\mathcal{M}(X, Y)$ when $Y \neq \phi$*

Given a couple (X, Y), let Σ' be an open orientable planar surface with $\beta\Sigma' \approx X$. If $Y \neq \phi$, it is easy to construct a sequence $(u_k)_{k \in \mathbb{N}}$ in Σ' whose set of accumulation points in the compactification by ends $\hat{\Sigma}'$ of Σ' is exactly Y.

Now identify Σ'_* with the leaf L_0 in the planar model $\mathcal{M}(X)$ provided by 3.2, 3.3 or 3.4. According to the definition of model, the sequence $(u_k)_{k \in \mathbb{N}}$ converges to $+\infty$ in $D^2 \times \mathbb{R}$, so the family $(n(u_k))_{k \in \mathbb{N}}$ of transversals is locally finite and we can, by making turbulizations on mutually disjoint tubular neighbourhoods of the $n(u_k)$ on the model $\mathcal{G}$, change the ends of L_0 corresponding to points of Y into nonplanar ends. This gives us the desired model $\mathcal{M}(X, Y)$.

5.2. *Construction of $\mathcal{F}_\Sigma$ when $\beta_1\Sigma \neq \phi$*

We proceed as in 3.5. Again there are two cases:

1) If Σ has an isolated end ε, we consider a model $\mathcal{M}(X, Y)$ for

$$(X, Y) \approx (\beta\Sigma - \{\varepsilon\}, \beta_1\Sigma) \text{ if } \varepsilon \text{ is planar, resp.}$$

$$(X, Y) \approx (\beta\Sigma - \{\varepsilon\}\ \beta_1\Sigma - \{\varepsilon\}) \text{ if } \varepsilon \text{ is nonplanar}$$

and take the restriction $\mathcal{F}$ of $\mathcal{M}(X, Y)$ to the interior of $D^2 \times \mathbb{R}$.

The desired foliation $\mathcal{F}_\Sigma$ will be $\mathcal{F}$ in the first case and is obtained from $\mathcal{F}$ in the second case by a construction as in 3.5: choose a sequence $(u_k)_{k \in \mathbb{N}}$ of points in the leaf L_0 of $\mathcal{M}(X, Y)$ which converges to a point of ∂L_0 and make turbulizations along the $n(u_k)$ on the model $\mathcal{G}$.

2) If $(\beta\Sigma, \beta_1\Sigma)$ is a couple of Cantor sets (C, C_1), we take the restriction of a model $\mathcal{M}(C, C_1)$ to the interior of $D^2 \times \mathbb{R}$. Exactly as in case 2) of 3.5 we kill off the isolated planar end introduced in L_0 by a sequence of turbulizations modelled on

$$\mathcal{M}(C, C) \text{ if } C_1 = C \text{ and on}$$

$$\mathcal{M}(C) \quad \text{if } C_1 \neq C$$

(the latter model being provided by 3.3).

This ends the proof of the main theorem. □

CONCLUDING REMARKS

i) All foliations constructed here are only C^∞. It is, however, not difficult to find an analytic atlas for them, using the fact that every leaf has a compatible analytic structure and Haefliger's theorem on the existence and uniqueness of a foliation in a neighbourhood of a proper leaf with prescribed holonomy ([H], p. 382–383).

ii) In view of the constructions presented here, it would not be difficult to construct a simple smooth foliation of $\mathbb{R}^3$ with countably many different types of leaves. However there remains the question whether there exists a (simple) foliation $\mathscr{F}$ of $\mathbb{R}^3$ such that any orientable open surface is diffeomorphic to some leaf of $\mathscr{F}$.

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ADDED IN PROOF:

At the time the proofs were corrected, the authors have found a way to simplify the constructions considerably (especially those in § 3). Moreover, they can prove that these foliations are defined by smooth submersions $\mathbb{R}^3 \to \mathbb{R}$.